Equations from Functions

Abstract

At the core of this note is the observation that, associated with every function F, there is a unique set of equations, one equation for each element Y_0 of the range of the function. Filling in the details of this observation brings out a clear, natural, and precise relationship that exists between a function and its related set of equations.

Functions

Here we establish terminology and symbolism for referring to functions.

A function is a mapping F from a *domain* set Dom(F) to a *range* set Ran(F). Given a function F, for every X in Dom(F) there is a unique element in Ran(F). This element is written as F(X). We say that F maps X to F(X), and write F: X ---> F(X)

The *image* set Ima(F) is the set of all F(X) for X in Dom(F). Ima(F) is a subset of Ran(F). If for every Y in Ima(F) there is a unique X in Dom(F) such that Y = F(X), then F is said to be 1-1 (injective).

Formally, a function is the set of ordered pairs (X, F(X)) for X in Dom(F). If we know this set of pairs, Dom(F) is the set of first elements and Ima(F) is the set of second elements.

(Once the formal characterization of a function F as the set of ordered pairs is complete, we have a choice for what we call Ran(F), as long as it includes Ima(F). In practice, a family of functions might each have different image sets, and specifying a simple set Ran(F) for the whole family helps give a simpler picture of the family.)

For many functions we are interested in, there is a rule or formula of some sort that allows us to find or compute $F(X_0)$ for every X_0 in Dom(F). But in the discussion that follows we don't care what this rule looks like. Here, F is just a name for the function, and $F(X_0)$ is the name for the element in Ima(F) that X_0 is mapped to by F.

Definition of "Equation based on a function"

Given a function F and an element Y_0 in Ran(F), we call

(1) $F(X) = Y_0$

an "equation based on F". A "solution" to an equation $F(X) = Y_0$ is an element X_0 in Dom (F) such that $F(X_0) = Y_0$.

Note that if Y_0 is in Ima(F), there must be at least one solution. If Y_0 is in Ima(F) and F is 1-1, there is exactly one solution.

Example 1:

Suppose s is the squaring function with Dom(s) and Ran(s) each the set of real numbers **R** and $s(x) = x^2$. Then Ima(s) is the non-negative reals. Here are three equations based on s:

(2a) $s(x) = 4$ (2b) $s(x) = 0$ (2c) $s(x) = -4$

The solutions of (2a) are 2 (since $s(2) = 2^2 = 4$), and -2 (since $s(-2) = (-2)^2 = 4$). There are no other solutions. The solution of (2b) is 0 (since $s(0) = 0^2 = 0$). There are no other solutions. There are no solutions at all to (2c), since $s(x) = x^2 \neq -4$ for any x in R.

Example 2:

Suppose S is the squaring function with Dom(S) and Ran(S) each the set of complex numbers **C** and $S(z) = z^2$. Then Ima(S) is also **C**. Here is an equation based on S:

(3) $S(z) = -4$

The solutions of (3) are 2i (since $S(2i) = (2i)^2 = -4$), and -2i (since $S(-2i) = (-2i)^2 = -4$). There are no other solutions.