D. Different Uses of Letters and Types of Representations in Algebra

Assigned to: Mike Oehrtman and Harris Shultz

Variables and Constants

A *variable* is a symbol which may change its value while it is under observation. Variables frequently are contrasted with constants: A *constant* is a number which is of interest, at least with respect to its current context. That said, we stress that what appears to be a variable and/or what appears to be a constant may in fact be the other depending upon the particular circumstance and the direction of the discussion. For example, in the formula

$$f(x) = ax^2$$

we are inclined to view x as the variable and a as a constant. As an illustration of this, it is standard practice in the study of quadratic functions to observe that if the constant a is positive then the graph of $f(x) = ax^2$ is a parabola opening upward, while if the constant a is negative then the graph opens downward. However, the study of parabolas frequently includes an exploration of the effect of a on the shape of the graph. Such an investigation reveals, among other things, that the effect of continuously changing a from say 0.5 to 3 is the narrowing of the parabola. So, in this setting, the symbol a serves as a variable.

In calculus we also encounter an ambiguity regarding which is the variable. In preparation for the study of derivatives we ask our students to compute and simplify a difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

for various functions *f*. As was the case above, the symbol x is naturally viewed as a variable. However, as we move to the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

the symbol h is clearly the variable and as h is approaching zero the symbol x certainly must be thought of as a constant.

Parameter

A *parameter* is a symbol which is assumed to be constant with respect to its current context but can assume different values as the situation changes. For example, in probability theory the mean or expected value of a random variable is a parameter usually denoted by μ. For that particular random variable the value of μ is constant. However, when a different random variable is introduced, its mean μ typically will have a different value.

In analytic geometry the general form for the equation of an ellipse centered at the origin having a horizontal major axis is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $a > b > 0$. The symbols a and b are parameters; they are constant for a given ellipse and, in fact, $2a$ and $2b$ are the lengths of the ellipse's major and minor axes, respectively. Clearly, the introduction of a new ellipse in general will be accompanied by new values for a and b.

Expression (written by William McCallum)

An *expression* gets its name from the fact that it expresses something. A mathematical expression expresses the result of a calculation with numbers. Some of the numbers might be explicitly given, like 2 or π or 1.25. Other numbers in the expression might be represented by letters, such as x, y, P , n, or ξ. The numbers these letters stand for might or might not be specified by the mathe- matical narrative surrounding the expression. The expression itself has nothing to say on this issue.

It is useful to think of expressions as sentence fragments, which can be used in many different sorts of sentences. Different types of sentences cause us to give different names to the letters. At any rate, no matter what the letters in an expression are called, it is understood that at any given moment when we are considering an expression, we are supposed to regard all the occurrences of the same letter in that expression as having the same value. Expressions have various syntax rules, which are captured in conventions about the order of operations and the use of parentheses.

The calculation an expression represents might use only a single operation, such as in $4 + 3$, or $3x$, or it might use a series of nested or parallel operations, such as in $3(a^2 + 9) - 9/b$. The expression might not have any operations at all. Both 3 and A are expressions. If the expression contains letters, then we do not know its value until we know the values of the letters. Choosing specific values for the letters and calculating the resulting value of an expression is called *evaluating the expression*.

Formula

A *formula* is a rule or fact expressed using mathematical symbols. For example, the formula $F = 1.8C + 32$ describes how to convert Celsius temperature to Fahrenheit temperature.

Coefficient

A *coefficient* is a constant that appears as a multiplicative factor of some object. For example, in the formula $F = 1.8C + 32$, the number 1.8 is the coefficient of C.

Quantifier

A *quantifier* is a logical symbol that indicates the quantity of a proposition. The two most common quantifiers in mathematics are $\forall$ (for all) and $\exists$ (there exists).

“For all”

- Makes a claim about all members of a particular set.
- In order to establish a universal statement, typically suppose that we are given an arbitrary element about which we know nothing other than it is a member of the set to which the universal statement is applied, then must show that the element has the given property.
- In order to establish a universal statement is false, we only need to provide a single counterexample.
- For an infinite set, no amount of verifying that the property holds for individual elements will suffice to prove the statement.
- The universal quantifier is sometimes suppressed (assumed from context)

“There exists”

- Claims that there is at least one element of a certain set satisfying a given property – an existential statement.
- To establish an existential statement
- There may be more than one element that satisfy the property. All elements of the set may satisfy the property.
- “such that…” - statement of a property that some element must satisfy

AN EXAMPLE

In the following example we solve a problem algebraically, specifically noting how we think about each variable at each step, then link that to the type of quantification that would formalize the statements. The problem is to find all lines through (a, a^2) intersecting the graph of $y = x^2$ exactly once, without any calculus. The rows list algebraic steps solving the problem and there is a column for each variable. Below each row is a symbolic expression of the step.

Using only algebra (no calculus), find the equations of all lines through the point $\left(a,a^2\right)$ intersecting the graph of $y=x^2$ exactly once.

	x	y	m	a	other
1. $y=x^2$	Variable (indep.) Input for parabola	Variable (dep. on x) Output for parabola			
$\forall x\in\mathbf{R},\exists y\in\mathbf{R}\ni y=x^2$					
2. $y=m(x-a)+a^2$	Variable (indep.) Input for line (different from step 1)	Variable (dep. on x) Output for line	Value to solve for. Constant slope for any given line. Variable slope to consider all lines.	Constant. Specific x-value. Can be any value.	
$\forall a\in\mathbf{R},\forall m\in\mathbf{R},\forall x\in\mathbf{R},\exists y\in\mathbf{R}\ni y=m(x-a)+a^2$					
3. $x^2=mx-ma+a^2$	Use the same value for both of the previous two interpretations. Any x where the graphs cross. A value to solve for.	Not in the equation but implicitly referenced. Heights/outputs for the two graphs are the same.	Constant slope. If considered variable, then the x's in column 1 change.	Constant.	
$\forall a\in\mathbf{R},\forall m\in\mathbf{R},\forall x\in\mathbf{R},\forall y\in\mathbf{R},y=m(x-a)+a^2\wedge y=x^2\Rightarrow x^2=m(x-a)+a^2$					
4. $x^2-mx+ma-a^2=0$ solve for x	Variable that makes an equation true for some values. A value to solve for.	N/A	Constant slope. Constant in an equation.	Constant.	
$\forall a,m\in\mathbf{R},\exists x\in\mathbf{R}\ni x^2-mx+ma-a^2=0$					
5. $x=\dfrac{m\pm\sqrt{m^2-4ma+4a^2}}{2}$	Specific values of x making equation 4 true.	N/A	Constant slope. Constant in a formula.	Constant.	Implicit constants: $a'=1,\quad b'=-m,$ $c'=ma-a^2$ 2 meanings for a
$\left(\forall a,m\in\mathbf{R},\exists x\in\mathbf{R}\ni x^2-mx+ma-a^2=0\right)\wedge\left(\forall a',b',c',x'\in\mathbf{R},a'x'^2+b'x'+c'=0\Rightarrow x'=\frac{-b'\pm\sqrt{b'^2-4a'c'}}{2a'}\right)\Rightarrow\left(x=\frac{m+\sqrt{m^2-4ma+4a^2}}{2}\vee x=\frac{m-\sqrt{m^2-4ma+4a^2}}{2}\right)$					
6. $m^2-4ma+4a^2=0$ b/c we want 1 solution	Not in the equation but implicitly referenced. A unique solution to equation 4.	N/A	Variable that makes an equation true for some values. A value to solve for.	Constant.	

$\forall a,m\in\mathbf{R},\left(\exists x\in\mathbf{R}\ni x=\frac{m+\sqrt{m^2-4ma+4a^2}}{2}\wedge x=\frac{m-\sqrt{m^2-4ma+4a^2}}{2}\right)\Rightarrow m^2-4ma+4a^2=0$					
7. $m=\dfrac{4a\pm\sqrt{16a^2-16a^2}}{2}$	N/A	N/A	Specific values of *m* making equation 6 true.	Constant.	Implicit constants: $a'=1, b'=-4a, c'=4a^2$ Implicit var: $x'=m$ 2 meanings for *a* & *x*
$\left(\forall a,\exists m\in\mathbf{R}\ni m^2-4ma+4a^2=0\right)\wedge\left(\forall a',b',c',x'\in\mathbf{R},a'x'^2+b'x'+c'=0\Rightarrow x'=\frac{-b'\pm\sqrt{b'^2-4a'c'}}{2a'}\right)\Rightarrow\left(m=\frac{4a+\sqrt{16a^2-16a^2}}{2}\vee m=\frac{4a-\sqrt{16a^2-16a^2}}{2}\right)$					
8. $m=2a$	N/A	N/A	Constant slope.	Constant.	
$\forall a\in\mathbf{R},\exists m\in\mathbf{R}\ni m=2a$					
9. $y=2a(x-a)+a^2$ $y=2ax-a^2$	Variable (indep.) Input for line	Variable (dep. on *x*) Output for line	Implicit: placeholder to substitute for.	Constant. Arbitrary.	
$\forall a\in\mathbf{R},\forall x\in\mathbf{R},\exists y\in\mathbf{R}\ni y=2a(x-a)+a^2$					
10. or $x=a$ vertical line also works	Potentially variable, but constrained to *a*.	Implicit: all real numbers (all heights)	N/A	Constant. Arbitrary.	
$\forall a\in\mathbf{R},\exists! y\in\mathbf{R}\ni y=a^2$					