Proportion

Jim Madden, July 2008

Dick Stanley has advanced the idea that proportionality may usefully be viewed as a relationship between real-valued functions. This perspective unifies many notions commonly arising in connection with proportion. This note elaborates and explains his contention as I interpret it.

1. A Proposal

Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}^\times$ denote the set of non-zero reals.

Definition. Let X be a set and let $f, g : X \to \mathbb{R}^\times$. We say that *$f$ and g are proportional* if there is a constant $k \in \mathbb{R}^\times$ such that

$$\text{for all } x \in X,\ g(x) = k\, f(x).$$

The number k is called the *constant of proportionality relating f to g.*

Remark. Some readers may prefer a narrower definition in which k is required to be strictly positive. We do not need this restriction in order to make the math work. (It might be possible to make the proposal work for functions that vanish at some points, as long as they don't vanish on all of X, but I did not investigate this.)

Examples.

1. Let X be the set of all circles in the plane. For any circle K, let $r(K)$ denote the (length of the) radius of K and let $c(K)$ denote the (length of the) circumference of K. Then r and c are functions from X to $\mathbb{R}$, and c is proportional to r because

$$\text{for all circles } K,\ c(K) = 2\,\pi\, r(K).$$

2. Let L and M be parallel lines, and let X be the set of all triangles with base on L and vertex on M. If T is a triangle in X, let $b(T)$ be the length of the base of T (measured in some standard unit) and let $A(T)$ be the area of T(measured in square units). If h is the distance from L to M, then

$$\text{for any triangle } T \in X,\ A(T) = \frac{h}{2}\, b(T).$$

   This is a restatement of Euclid's *Elements*, Book VI, Proposition 1: *Triangles and parallelograms which are under the same height are to one another as their bases.*

3. Let $\mathbb{A}$ be the Cartesian plane with coordinate functions x and y. By x we understand the function from $\mathbb{A}$ to $\mathbb{R}$ whose value at the point P (denoted $x(P)$) is the x-coordiante of P. Similarly $y(P)$ denotes the y-coordinate of P. Let L be the line through the point $P_{(2,7)}$ and the origin. Then,

$$\text{for any point } P \in L,\ y(P) = \frac{7}{2}\, x(P).$$

   Thus, the restrictions to L of x and y are proportional. Similarly, if K is the line through $P_{(1,-3)}$ and origin, then for all P on K, $y(P) = -3\, x(P)$. In this case, -3 is the constant of proportionality relating the functions $x|_K, y|_K \colon K \to \mathbb{R}^\times$.

4. Let X be the set of all nuggets of pure gold that are available for measurement. For any nugget x, let $w(x)$ be the weight of the nugget measured in grams and let $V(x)$ be volume of x measured in cubic centimeters. Then

$$\text{if } x \text{ is any nugget in } X,\ w(x) = 19.3V(x).$$

   The weight of a nugget of pure gold is proportional to its volume. The constant of proportionality is 19.3, the density of gold in grams per cubic centimeter.

2. Ancient and Modern Ideas about Proportion

In the *Elements*, Book V, Euclid gives the following

> *Definition 3. A ratio is a sort of relation in respect of size between two magnitudes of the same kind.*
>
> *Definition 4. Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another.*

Definition 5 states Eudoxus' criterion for equality of ratios. Proportion is introduced in the next definition:

> *Definition 6. Magnitudes which have the same ratio are called* proportional.

In the typical schoolbook problems, three terms of a proportion are given and the fourth is sought:

> *If 3 oz. of Silver cost 17s. what will 48 oz. cost?*[1]
>
> *If a man can travel 400 miles in 15 days, how far can he travel in 9 days?*[2]

Euclid's definitions are echoed in many elementary modern treatments of ratio and proportion. For example, one web site ostensibly geared to school math contains the following statements:

> *A ratio is a comparison of two numbers. To compare ratios, write them as fractions. The ratios are equal if they are equal when written as fractions. A proportion is an equation with a ratio on each side. It is a statement that two ratios are equal.*[3]

In more-advanced contexts, proportion tends to be treated differently. The *Wikipedia* entry on proportion begins:

> *In mathematics, two quantities are called proportional if they vary in such a way that one of the quantities is a constant multiple of the other, or equivalently if they have a constant ratio. ... Given two variables x and y, y is (directly) proportional to x if there is a non-zero constant k such that $y = k\,x$.*

The *Wikipedia* definition appears close the one we are proposing. However, it does not explicitly address the issue of why or how the two variables, x and y are linked. In the second sentence, there is an asymmetry in roles of the variables. They appear to be linked by a functional relationship. The idea that proportionality is a special kind of functional relationship is explicit in some discussions. An essay available at the Dana Center web site suggests that students may

> *think of proportional relationships in terms of functions. Specifically, writing $y = k\,x$ is a way students can begin to express the relationship between two proportional quantities y and x using functions: the quantity y is proportional to, or depends on, or can be determined from, or is a function of, the quantity x.*[4]

The proposal we are advancing goes in a different direction, however. Though functions play a role in the statement of our proposal, the quantities that are related in a proportion need not be bound to one another by an input-output relation. We will elaborate below.

3. Contrasts and Conflicts

There is an apparent gulf between the ancient definition (and its contemporary manifestations in school math) and the more modern conceptualization exemplified in the *Wikipedia* article. In the first place, the ancient conception of proportion is specific while the modern conception is generic. In the ancient framework, a proportion is a relation between four fixed magnitudes appearing in two ratios. There are

[1] This is the first problem in the section entitled "Of the Single Rule of Three" in Dilworth's *Schoolmaster's Assistant*. This problem and several more from the same source are copied and solved in the surviving pages of Abraham Lincoln's school sum book, written in his mid-teens.

[2] From *Practical Arithmetic* by James Bates Thomson. (1864).

[3] `www.mathleague.com/help/ratio/ratio.htm`

[4] `http://www.utdanacenter.org/mathtoolkit/downloads/support/proportionality.pdf`

no variable quantities nor any explicit reference to an enduring relationship between kinds. If the ancient conceptualization captures any generality, it is in the idea that similar valid proportions might be written about other specifics. A modern statement of proportionality, on the other hand, speaks explicitly of two general things (each with numerous instantiations) and of an enduring relationship (that is manifest in each instantiation).

Second, in the modern perspective, there is an ontological shift, a change in objects of attention. A proportion is no longer a relationship between two *ratios*, but a relationship between two *variables*. Ancient proportion is a comparison between comparisons of objects of direct experience. Modern proportion is a comparison between two generic objects. Here, the connection to direct experience is not via specific comparisons of concrete objects, but in the idea that the generic object has instances that are concrete.

A third major change is in kind of ratios that appear in the ancient and modern conceptualizations. For the Greeks, ratios were direct comparisons of lengths to lengths, areas to areas, counting number to counting number, *etc.* For a variety of good reasons internal to ancient mathematics, ratios across kinds were not made. In conformity with this tradition, schoolbooks through the 19^{th} century and well into the 20^{th} emphasized that ratios may be formed only between like quantities. A well-formed proportion might equate a ratio of prices to a ratio of weights or a ratio of distances to a ratio of times, but explicitly stated conventions prohibited forming a ratio of price to weight or of distance to time. (This does not mean that people did not calculate and use such things; all I am saying is that the textbooks of the 19^{th} century obsessively repeated the ancient Greek strictures.) In contrast, the first sentence of the *Wikipedia* excerpt mentions a ratio between two varying quantities of *different* kinds. As a matter of fact, constants of proportionality are almost always ratios across species. A concern for the ancient tradition has led many writers to insist that a constant of proportionality should not be called a "ratio" at all, but a "rate".

The contrast between the ancient and modern treatments is vivid in the second example in the first section. Euclid would say: Given two triangles with the same height, the ratio of their areas is equal to the ratio of their bases. The author of the *Wikipedia* article would say that if A is the area of a triangle and b is its base, then for all triangles with height h, $A = (h/2)b$. Now, the question we seek to answer is, does the difference we see here imply a change in the meaning of proportion? Have we transitioned to a fundamentally new relationship? Or are the ancient and modern conceptions linked at some deep level?

4. Reconciliation

Despite the differences we have noted, the ancient and modern conceptual constructs have in common the idea of a relationship that persists from one situation to another. We figure the price of 48 oz. of silver assuming that the relationship of price to weight does not vary from the original situation where 3 oz. were obtained for 17s. We assume that the man travels at the same rate on all trips, so if he doubles his distance he doubles his time, if he triples his distance he triples his time, and so forth, and conversely when multiplying his time by any chosen factor he multiplies his distance by the same factor. Even in the ancient setting, a proportion is a means of extrapolating from one situation to another. The price of any quantity of silver can be computed, provided the same market prevails. The distance travelled in any number of days can be found, provided the same man travels.

The notion that there is something that endures and exerts a uniform influence on the relationships between the specific magnitudes observed on different occasions is common to both the ancient and modern conceptions. To capture the idea of something that is general but that has specific instances, we might think of a species or a kind: all circles, all triangles with a given trait, all points on a given line, all nuggets of gold, all sales of silver in a given market, all walking trips by a man who always walks at the same rate. Within each species, there are individual instances and each instance has specific quantitative aspects. Proportionality refers to the way these aspects relate from instance to instance.

To investigate the properties of proportional relationships, we will build a model that enables us to experiment with idea of a species with instances, each of which has particular quantitative aspects. We want to reduce to the bare essentials. We need a collection of things and we the quantities associated with each of at least two aspects of each thing. Modern mathematics supplies what we need to build a model of extraordinary generality from meager resources. A set is a collection of individuals between whom we can distinguish, but about whom we need know nothing else. So imagine a set X that has at least two elements; the set is the species, and the elements are the instances. As yet, our elements have no quantitive aspects or associated numbers. To complete our model, then, we need to equip each element of X with these things. The minimal way to do this is simply to associate numbers to the element of X, one number for each aspect. The mathematical object that does such work is a function. So, imagine two functions, f and g from X to $\mathbb{R}^{\times}$. Each function tells us the value of a specific quantitative aspect of the individual to whom we apply the function. If $x \in X$, $f(x)$ is the magnitude of the "f-ness" of x and $g(x)$ is the magnitude of the "g-ness" of x.

By taking a set equipped with two such functions, we build an imaginary model of all the data we absolutely require to speak about proportionality. But we have not guaranteed that there is a proportional relationship. So let us perform a test to see whether proportionality is present. The ancients would make the test by asking if it is it the case that for any two elements of X, the ratio of their f-values is the same as the ratio of their g-values? In a more modern paraphrase, is it the case that the following *ancient criterion for proportionality* holds?

$$\text{For all } z, w \in X,\ \frac{f(z)}{f(w)} = \frac{g(z)}{g(w)}\ . \qquad (ACP)$$

If APC is true, then we have a situation that can be described in perfect analogy to the phrasing of *Elements* VI,1, *viz.*, *The objects in X have f aspects that are to one another as their g aspects.* Moderns, on the other hand, would make the test by asking whether the following *modern criterion for proportionality* is true:

$$\text{There is } k \in \mathbb{R}^{\times} \text{ such that for all } x \in X,\ g(x) = k\, f(x). \qquad (MCP)$$

As a matter of fact, the two tests are equivalent:

Proposition. *For any set X and functions $f, g : X \to \mathbb{R}^{\times}$, (ACP) is true if and only if (MCP) is.*

Proof. Suppose $f, g : X \to \mathbb{R}^{\times}$ satisfies (APC). Select a specific element $w_0 \in X$. Then for all $x \in X$, $g(x) = \frac{g(w_0)}{f(w_0)} f(x)$. Let $k = \frac{g(w_0)}{f(w_0)}$. This shows (MCP). Conversely, suppose (MCP) is true. Pick any $z, w \in X$. Then $\frac{g(z)}{g(w)} = \frac{k\, f(z)}{k\, f(w)} = \frac{f(z)}{f(w)}$. QED

5. Conclusions

The essential features of situations where proportions arise are modeled by an abstract system consisting of a set X (with two or more elements) equipped a pair of $\mathbb{R}^{\times}$-valued functions whose values have a constant ratio. We call such a system a *model of proportionality*. In specific situations where proportional relationships arise, the set X is a collection of objects belonging to some natural kind (*e.g.*, circles, triangles of some specific type or other geometric or mathematical objects, or other kinds of things such as economic transactions, idealized physical objects, *etc.*), and the functions are quantitative aspects of the individuals in the kind (*e.g.*, the radius, diameter or circumference of a circle or the side lengths, areas or angles of a triangle, or the price paid and amount purchased in a transaction, *etc.*). Some important features of proportional situations are not immediately evident in the tools used to reason about them. The analogical relations that appear in Euclid and in schoolbook exercises in proportion do not make explicit the enduring conditions that justify the analogy. Similarly, the contention that proportionality is a homogeneous linear relation between two "varying quantities" also leaves the out the background that explains the relation. Of course, the model of proportionality that we propose does not itself provide these missing components, but it does give a conceptual framework that has within it a place for this information to be displayed in a uniform way in many different examples.

6. Extensions

Elements, Book V, Definition 12 defines the alternation of the proportion $A : B :: C : D$ to be the proportion $A : C :: B : D$. Proposition 16 says that if four magnitudes are proportional, then they are also proportional alternately. As usual, the proof begins with a restatement of the claim in terms of labelled quantities. It reads, "*Let A, B, C, and D be four proportional magnitudes, so that A is to B as C is to D. I say that they are also so alternately, that is A is to C as B is to D.*"

In terms of the model we have proposed, Proposition 16 says that if f_1 and f_2 are proportional functions on $X = \{1, 2\}$ then the functions $g_1, g_2 : \{1, 2\} \to \mathbb{R}^\times$ defined by $g_i(j) := f_j(i)$ are also proportional. We will extend Proposition 16 to a more general alternation principle, suggested by the following design:

$$\begin{array}{cccc} a\,p & a\,q & \cdots & a\,r \\ b\,p & b\,q & \cdots & b\,r \\ \vdots & \vdots & \vdots & \vdots \\ c\,p & c\,q & \ldots & c\,r \end{array}$$

Proportionality, as we have defined it, is an equivalence relation on the set of functions on a set. In the design above, each rows may be viewed as function defined on the columns and each column may be viewed as a function defined on the rows. The entries make it clear that if the rows are in the same proportionality class, then the columns are, too. More abstractly, we have the

Alternation Theorem. *Let $\{ f_j \mid j \in J \}$ be a family of functions f_j from a set X to $\mathbb{R}^\times$, all in the same proportionality class. For each $x \in X$, define a function g_x from J to $\mathbb{R}^\times$ by*

$$g_x(j) := f_j(x).$$

Then the functions $\{ g_x \mid x \in X \}$ are in the same proportionality class of functions from J to $\mathbb{R}^\times$.

Proof:. Pick one of the f_j and call it F. Then, for each fixed j, we have a fixed non-zero real number k_j such that for all $x \in X$, $f_j(x) = k_j\, F(x)$. Now, for any two elements $z, w \in X$, we have: for all $j \in J$:

$$\begin{aligned} g_z(j) &= f_j(z) \\ &= k_j\, F(z) \\ &= k_j\, [F(z)/F(w)]\, F(w) \\ &= [F(z)/F(w)]\, f_j(w) \\ &= [F(z)/F(w)]\, g_w(j). \end{aligned}$$

Thus, g_z and g_w are in the same proportionality class of functions on J. QED