What do we mean by proportionality?

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The structure of this talk

- Part 1 will include
  - a mathematical view of ratios and proportional relationships, not intended for students in grades 6 and 7, but for the mathematical community, and
  - some thoughts about how these ideas develop beyond these grades, giving a rationale for this approach.
- Part 2 will be about some ideas for how this could translate into work on curriculum.

Part 1

Things that need definitions

- Ratio
- Equivalent ratios
- Proportional relationship
- Rate

Some definitions that underlie 6-7

- Def 1. A **ratio** is an ordered pair of positive rational numbers denoted $a : b$.
  - When asking, "What is the ratio of flies to frogs when there are 15 flies and 3 frogs," the answer is 15:3.
- Def 2. $a : b$ is **equivalent** to $c : d$ iff there exists an $s > 0$ s.t. $c = sa$ and $d = sb$.
  - Note that this is an equivalence relation.

(Remember—these defs are not for students.)

Some definitions that underlie 6-7

- Note 1. An equivalence class for this equivalence relation is called a **ratio relationship** in grade 6 and a **proportional relationship** in grade 7. There is a reason for this.
- Note 2. A **rate** is an element of the corresponding quotient space, but only when *interpreted in a context*.
  - 15 flies for every 3 frogs is the same rate as 5 flies for every 1 frog

Their geometry

- A ratio is an ordered pair in the first quadrant.
- Equivalent ratios lie along a line through (0,0).
- Equivalence classes are rays through (0,0) in the first quadrant.
- A rate is an interpretation in a context of a point on the projective line.

Some more definitions

- Def 3. The **value of a ratio** $a : b$ is a/b.
- Note 3. When a ratio represents a context, a **unit rate** is the amount of one quantity needed for 1 unit of the other quantity to form an equivalent ratio.
  - Note that a unit rate can be thought of as an *interpretation* of the value of a ratio in a context.

(Remember—these defs are not for students.)

Some theorems that underlie 6-7

- Thm 1. If two ratios are equivalent, their values are equal.
  - This implies that the value of a ratio is an invariant of its equivalence class.
- Thm 2. An equivalence class can be represented by an equation of the form $y = kx$, where $k > 0$ and $x \geq 0$.
  - Note that when we represent them with equations and graphs systematically in grade 7, we start calling them proportional relationships.

Grade 8 and HS

- Because the value of a ratio is an invariant of its equivalence class, we begin to abuse language:
  - The term *ratio* gets used to mean a ratio (15:3), any equivalent ratio (5:1), or its value (5/1).
  - This can the source of endless confusion for students!

Grade 8 and HS

- The term *rate* gets used to mean a unit rate together with a rate unit, like 45 km/hr.
- This shift is very sophisticated. We go from talking about a rate as an interpretation of a point in projective space to a symbolic entity that can be manipulated like a number.
  - "A car traveled 90 miles in 2 hours. At this rate, how far would it travel in 4 hours?"

Why proportional relationships?

- “Setting up a proportion” is a one-trick pony.
  - Students often learn to do it without understanding, so make many mistakes and cannot reason about problems that do not fit a very narrow type.
  - One can always understand proportions if one understands the grade-appropriate versions of the defs and thms laid out here, but not the other way around.

Why proportional relationships?

- Proportional relationships naturally lead to special linear relationships, which lead to linear relationships.
- All of this work has a big payoff, and provides a pivot point from arithmetic in elementary school to the study of relations and functions in grade 8 and high school.

Why spend so much time on thinking about definitions?

- The concept of equivalence is hard at all levels even when it seems obvious to experts. We are in too much of a rush to get to the punch line, and give short shrift to the ideas students should grapple with on the way there.

Part 2

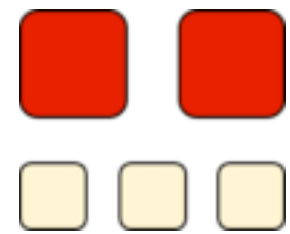
Reality check: grade 6 students

- In grades K-5, students represent quantities with increasingly abstract representations:
  - Objects
  - Literal pictures
  - Discrete pictures
  - Tape diagrams
  - Number lines

Using language and context to understand Ratios

- Using ratio and rate language in natural contexts
  - First just the ratio
  - Later, use the language to help students understand what we mean by equivalent ratios

A recipe calls for 2 cups of tomato sauce and 3 tablespoons of oil. We can say that the ratio of cups of tomato sauce to tablespoons of oil in the recipe is 2:3, or we can say the ratio of tablespoons of oil to cups of tomato sauce is 3:2.

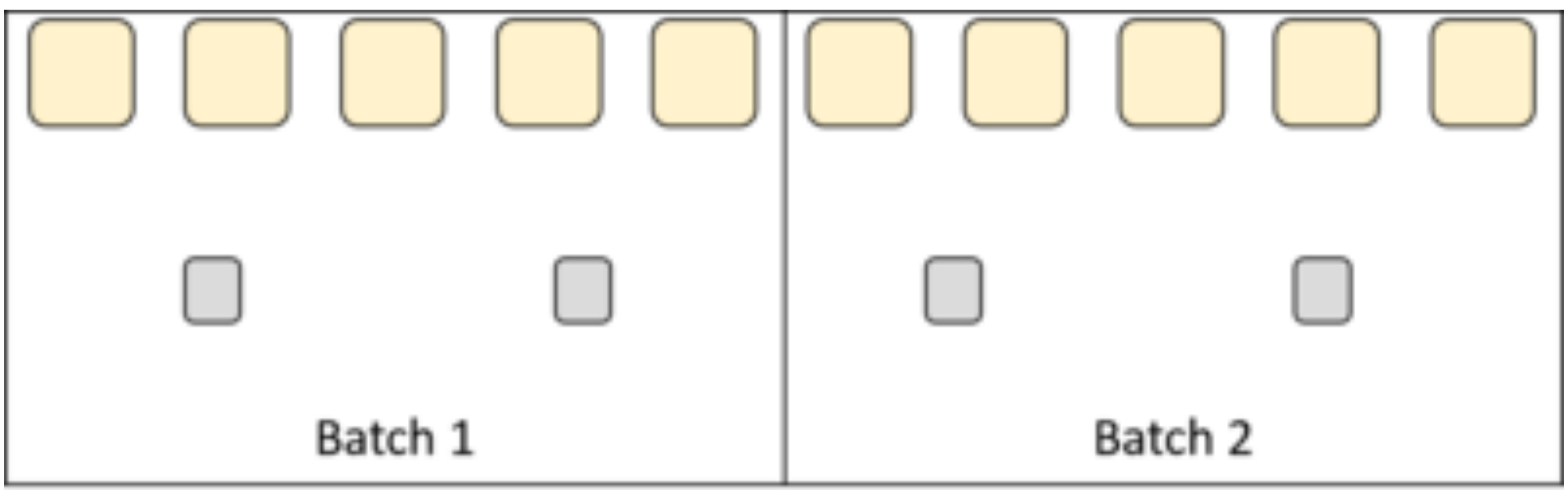


For each of the following situations, draw a diagram and name two ratios that represent the situation.

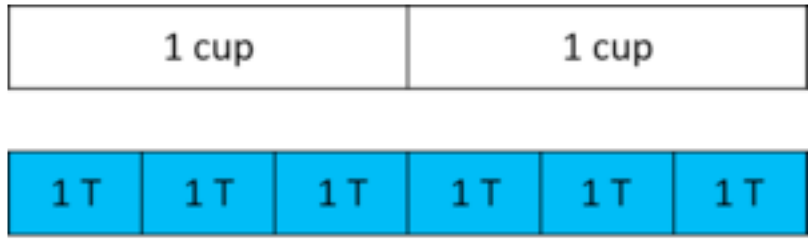
- To make papier-mâché paste, mix 2 parts of water with 1 part of flour.
- A farm is selling 3 pounds of peaches for $5.
- A person walks 6 miles in 2 hours.

Batch reasoning

- Diagrams can be used to show the multiplicative structure
  - The number of batches represent scale factors that later get abstracted as the scale factor in *sa* : *sb*



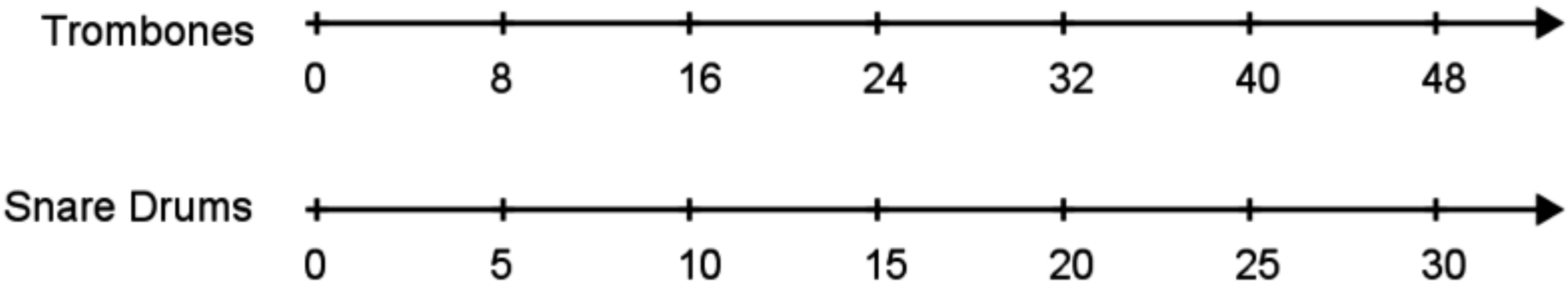
Abigail mixed 2 cups of white paint with 6 tablespoons of blue paint.



Which of the following statements describes this situation?

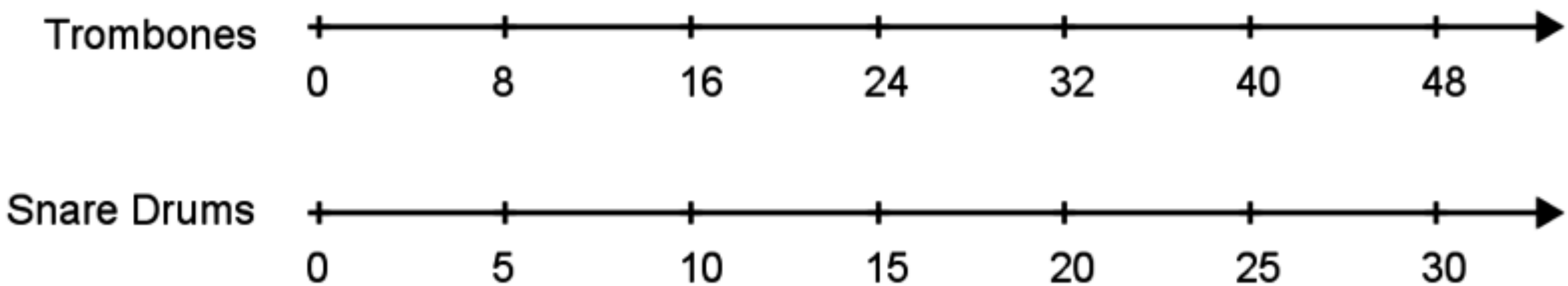
- There is 1 cup of white paint for every 3 tablespoons of blue paint.
- The ratio of the number of cups of white paint to the number of tablespoons of blue paint is 1:3.
- There are 3 tablespoons of blue paint per cup of white paint.
- For each tablespoon of blue paint there are 3 cups of white paint.

Double number lines and ratio tables



Trombones	24	8	32	64
Snare Drums	15	5	20	40

Double number lines and ratio tables



Trombones	24	8	32	64
Snare Drums	15	5	20	40
T per SD	1.6	1.6	1.6	1.6

Tying it all together

- Tables are very useful for seeing that the values of equivalent ratios are equal (Thm 1)
- Once students have internalized that equivalent ratios are of the form $a : b$ and $sa : sb$, then an argument like $a/b = sa/sb$ might be appropriate for students.
  - It is an empirical question how important it is to generalize at this level for students, one for which I have no data.

Tying it all together

- A tables of ratios for quantities x and y equivalent to $a : b$ can be used to see that $y = b/a\ x$. (Thm 2)
- Sophisticated students might benefit from seeing the argument that since

$$y/x = b/a, \text{ then}$$
$$y = b/a\ x.$$

  - But then again, maybe not!

In summary

- The mathematical community needs to get it straight what we are talking about when we talk about the mathematics of the K-12 curriculum.
- Once we are clear ourselves, we have to figure out how to use this scaffolding to create learning experiences for students that make sense and get them to the appropriate level of understanding for their grade level.