An Historical Perspective of Proportion, Ratio and Measurement

James J. Madden
Louisiana State University

Essential Mathematical Structures and Practices in K–12 Mathematics
AMS Special Session at the Joint Mathematics Meeting
Seattle, January 7, 2016

Main assertions of this talk in "history of ideas"

1. Classical (ancient Greek) mathematics includes a *category of being* distinct from number, called **magnitude**. Magnitudes come in kinds, and we can perform operations on the magnitudes of a given kind without any reference to numbers.
2. Modern-Practical mathematics (with Hindu-Arabic roots) developed between 800 and 1600 AD, propelled by applications in commerce and trade. This mathematics takes number as the principle quantitative category. School textbooks tend to use the Modern-Practical ontology, but many try to mix in Classical terminology, sometimes with grotesque results.
3. Wallis and Newton understood ratio as a function from pairs of magnitudes (of a given kind) to numbers, and were explicit on this. Their point of view successfully reconciles the Classical and the Modern-Practical perspectives.

Main assertions of this talk related to math instruction

1. The Common Core State Standards
   a) use the word "quantity" without a precise meaning; it is not clear if CCSS admits quantities that are **not** numbers (or numbers-with-unit-labels);
   b) treat measurement classically in Grades K-5, but do not take advantage of the opportunities this creates for treating ratio, proportion, dimensional analysis and modeling later on.
2. School mathematics would benefit by having a clear, explicit ontology that:
   - is embraced by mathematicians,
   - meshes with the way math is used in the sciences,
   - connects measurement meaningfully to ratio and proportion,
   - explains "dimensional analysis" (units, unit cancellations) and
   - provides optimal support for reasoning in modeling situations.
3. Euclid, Wallis, Newton and Hölder solved this problem in the scholarly arena. We can apply their ideas productively in engineering the K-12 curriculum.

Historical Time Lines

Classical Frame (Euclidean Ratio)

Aristotle (384 – 322 BC)
Euclid (fl.300 BC),
Elements, Book V
Boethius (480 – 524 AD),
De arithmetica, (circa 505)
Thomas Bradwardine (c.1290 – 1349),
Tractatus de proportionibus (1328)
Galileo Galilei (1564 – 1642)
Johannes Kepler (1571 – 1630)
John Wallis (1616 – 1703)
Isaac Newton (1642 – 1727)
James Stewart (1941 – 2014)
Calculus (1987)

Practical-Modern Frame (Rule of Three)

Aristotle (384 – 322 BC)

Muḥammad ibn Mūsā al-Khwārizmī (c.780 – c.850)

Leonardo Fibonacci (c.1170 – c.1250),

Liber Abaci (1202)

Michael of Rhodes (c.1380 – c.1450)

Robert Recorde (c.1512 – 1558)

The Ground of Artes (1543)

Edward Cocker (1631 – 1676),

Arithmetick (1677)

Thomas Dilworth (c.1705 – 1780),

Schoolmaster's Assistant (1740)

James Stewart (1941 – 2014)

Calculus (1987)

In the next few slides,
we view some passages illustrating the
Modern-Practical Frame,
in the context of the so-called Rule of Three.

Note the focus on the classification and manipulation of numerical data.

Cocker and Dilworth (slides 3 and 4) show classical influence.

Klapper (slide 5) shows modern algebra influence.

1. al-Khwārizmī (c. 810)

"Know that all transactions between people, be they sales, purchases, exchange, hire, or any others, take place accoring to two modes, and accoring to four numbers pronounced by the enquirer: the evaluated quantity, the rate, the price, and the evaluated quantity. . . . [A]mong these four numbers, three are always obvious and known, and one of them is unknown . . . You examine the three obvious numbers. Among them it is necessary that there be two, of which each is not proportional to its associate. You multiply [them] and divide the product by the other obvious number . . .; what you get is the unknown number sought . . ."

Source: al-Khwārizmī: *Treatise on Algebra*, Chapter on transactions. Translated by Roshdi Rasheed, 2009.

2. Fibonaci (1202)

> "Four numbers are found in these negotiations, of which three are known and one is unknown. The first is the number of items sold, or the weight or measure of the sale: a hundred hides or goatskins, or a hundredweight or a hundredpound, or pounds, or ounces, or pints of oil, or sestarios of corn, or bundles of cloth. The second is the price of the sale: a quantity of denari, or of bezants, or of tareni . . . The third is another quantity of the same merchandise, and the fourth is the unknown price [to be determined]."

Source: L. Sigler. *Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation*. Springer, New York, 2012. (Chapter 8)

Problem. *If 12 hides cost 30 denari, what will 42 hides cost?*
Solution (by Fibonacci's "Principal Method"). Make a square as follows:

12	42
30	?

Multiply the two numbers that lie in the ascending diagonal and then divide by the remaining number. The result is the price to be paid.

3. Cocker (1677)

> "The Relation of Numbers in Quality (otherwise called Proportion) is the Reference or Respect that the Reason of Numbers have one unto another; therefore the Terms ought to be more than two. Now the Proportion . . . is either Arithmetical of Geometrical."

> "The Rule of Three . . . is that by which we find out a fourth Number in Proportion unto three given Numbers, so as this fourth Number that is sought may bear the same Rate, Reason, and Proportion to the third (given) Number as the second doth to the first . . ."

> ". . . to find out the fourth Number in Proportion, . . . multiply the second Number by the third, and divide the Product thereof by the first . . ."

Source: *Cocker's Arithmetick*, 48th ed., London, 1736. Viewed on Google Books.

4. Dilworth (1740)

"Q. *By what is the* Single Rule *of* Three *known?*

A. By *three Terms*, which are always given in the Question to find a *Fourth*.

...

Q. *What do you observe concerning the* first *and* third Terms?

A. They must be of the same Name and Kind.

Q. *What do you observe concerning the* fourth Term?

A. It must be of the same Name and Kind with the *Second*.

...

Q. *How is the fourth* Term *in* Direct Proportion *found?*

A. *By multiplying the second and third Terms together and dividing that Product by the first Term."*

Source: T. Dilworth, *The Schoolmasters Assitant*, Henry Kent, 1752. Viewed on Google Books.

5. Klapper (1921)

> "A proportion is merely one method of writing a simple equation, and with the use of the letter x allowed, the equation form is likely to replace that of proportion. ... For example, consider this problem: If a shrub 4 ft. high casts a shadow 6 ft. long at a time that a tree casts one 54 ft. long, how high is the tree? Here we may write a proportion in the form $6\text{ ft.} : 4\text{ ft.} = 54\text{ ft.} : (?)$, not attempting to explain it, but applying only an arbitrary rule. This is the old plan. Or we may put the work into equation form,
>
> $$\frac{x}{54} = \frac{4}{6},$$
>
> and deduce the rule for dividing the product of the means by the given extreme ..."

Source: Paul Klapper. *The Teaching of Arithmetic: A Manual for Teachers*. D. Appleton, New York, 1921. Page 183.

Classical Magnitude

Magnitude: A basic ontological category

- The Euclidean theory of ratio is built on the more fundamental notion of **magnitude**.
- *Elements* does not spell out explicitly what a magnitude is, but we can infer much about what Euclid meant by magnitude from *Elements*, Book V.
- Axioms for Magnitude were formulated by Hölder in 1901, based on Book V.

Discrete versus continuous quantity in Greek thought

In *Categories*, Aristotle says:

> "Quantity is either discrete or continuous. ... Instances of discrete quantities are number and speech; of continuous, lines, surfaces, solids, and, besides these, time and place."

A magnitude, as understood by Euclid, was an individual belonging to a continuous quantitative kind, e.g., a line segment (belonging to the kind "line"), a polygonal region (belonging to the kind "surface"), a 3-dimensional body (belonging to the kind "solid").

Euclidean magnitudes are idealized objects of experience

- The objects of ancient Greek mathematics were idealized versions of things that one might see and touch and the manipulations that one might perform with them.
- Greek geometry was not "arithmetized." Today, we almost automatically connect numbers to lengths or areas. There was no hint of any such connection in Greek mathematics.

Idealized objects of experience (cont.)

David Fowler, in his book *The Mathematics of Plato's Academy: A New Reconstruction*, page 20, writes:

> "Greek mathematicians seemed to confront directly the objects with which they were concerned: their geometry dealt with the features of geometrical thought experiments in which figures were drawn and manipulated, and their arithmetic concerned itself ultimately with the evident properties of numbered collections of objects. Unlike the mathematics of today, there was no elaborate conceptual machinery, other than natural language, interposed between the mathematician and his problem."

Newton interpreted magnitude in the Euclidean sense:

> "*Magnitudes are said to be equal, which being placed one upon the other, are, or seem to be, congruous*; as *Lines*, *Angles*, *Surfaces*, which being compared by mental Apposition, are seen upon Account of some given Circumstances to coincide; and *Solids*, which penetrate each other, and coalesce into one. But *Magnitudes, so named in a looser Sense* because they can be increased or diminished, *are said to be equal, which considered as Causes, produce the same Effects*; as the *Times*, in which a Body moved uniformly is carried over equal Spaces; the *Velocities*, with which Bodies moved forward are carried over equal Spaces in a given Time; and *Forces*, which when they are opposed destroy each other."

Source: *Of the Measures of Ratios*, translated by J. Maguire, appendix to *Universal Arithmetick*, London, W. Johnston, 1769. Italics original.

We experience magnitudes at various levels of abstraction. . .

1. (Grades K-2) A thing that has a continuous quantitative attribute, such as length or weight; e.g., a pencil (a length), a lump of clay (a weight). (Other kinds—area, volume, duration—come in later grades.)

2. (Euclid) An ideal entity, possessing a continuous quantitative attribute; e.g., a line segment (a length), a polygon (an area).

3. (Physics) Physical lengths, masses or durations. They may be compared, added and multiplied by numbers. (Other, more complex magnitudes arise, but we pass over them for now.)

4. (Hölder) The equivalence classes of the magnitudes of a given kind form an archimedean totally-ordered group, and every such group isomorphic to a subgroup of the additive reals.

5. (Mathematics) A real number.

...and learn much about them before mastering arithmetic

Magnitude concepts develop in parallel with—and initially independent of—the acquisition of numerical skills:

- By age 5, children are able to identify measurable attributes (length, weight), to compare things with respect to attribute and to use representations to make indirect comparisons.
- After this, they acquire the ability to put several things in order with respect to a measurable attribute that they all share, and to build up varying lengths by laying units end-to-end (or varying weights by combining weights).
- Up to about age 7 (second grade) the ability to compare and add are elaborated and refined, while the idea of using a number of identical units to represent an arbitrary length (or weight) begins to develop.

Source: Sarama & Clements. *Early Childhood Mathematics Education Research: Leaning Trajectories for Young Children*. Routledge, New York and London, 2009.

Relations and Operations on Magnitudes

Euclid performs the following actions with magnitudes:

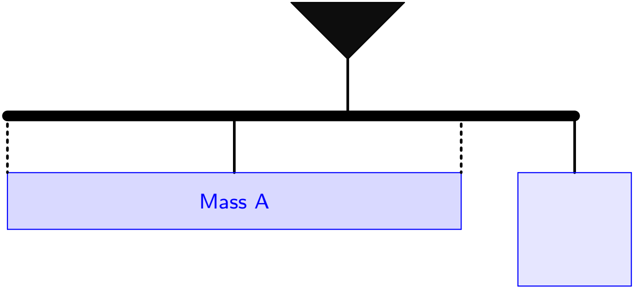
1. **Compare.** Given two magnitudes of a kind, we can detect which is larger, or see that they are equivalent.
2. **Add.** Given two magnitudes of a kind, we can add them to make a larger thing of the same kind, e.g., put two lengths end-to-end to form a new length.
3. **Duplicate and multiply.** We can make copies of a magnitude, and we may add a given magnitude A to itself over and over to form a multiple nA of it.
4. **Subtract.** A smaller magnitude may be removed from larger one of the same kind.
5. **Form a Ratio.** Two magnitudes A and B determine all comparisons of all multiples mA and nB. (See next section.)

Axioms for Magnitude

Euclid assumes:

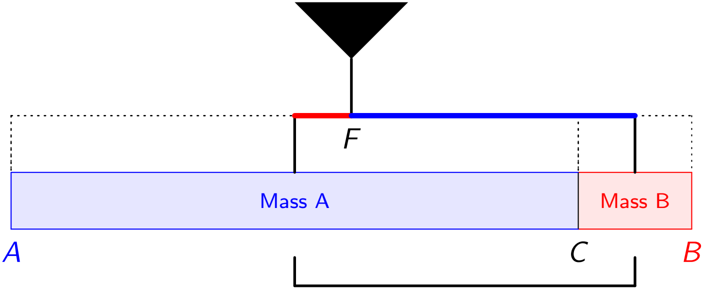
1. The operation of addition is not sensitive to the order in which the parts are joined or assembled (i.e., it is associative and commutative).
2. Addition of the same magnitude to two others preserves order. In other words, if A is less than B, then $A + C$ is less than $B + C$. The same is true of subtraction; if A is less than B, then $A - C$ is less than $B - C$. If A is equivalent to B and the same magnitude is added to (or subtracted from) both then the resulting magnitudes are equivalent .
3. *Archimedean Axiom.* Given a lesser and a greater magnitude, some multiple of the lesser exceeds the greater. (Book V, Definition 4: "Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another.")

Reasoning with Magnitudes: Galileo, *Two New Sciences*



Assumption: Suppose a beam is suspended by a string, and several masses are suspended from the beam. Suppose the beam balances when **uniform** Mass A is suspended from its ends. Then the beam will remain in balance when Mass A is suspended from its center.

Reasoning with Magnitudes: Galileo, *Two New Sciences*



$$\frac{1}{2}AB = \frac{1}{2}AC + \frac{1}{2}CB$$

Law of Levers. If Mass A and Mass B are suspended from a beam, then they balance when the ratio of the distances from the suspension points of Mass A and Mass B to the fulcrum F is equal to the ratio of Mass B to Mass A.

Euclidean Measurement/Ratio According to Hölder

Main Source:

Otto Hölder, Die Axiome der Quantität und die Lehre vom Mass, *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physikaliche Classe* 53 (1901), 1-64.

English translation in *J. Math. Psychology* 40 (1996).

Both Wallis and Newton say: "A ratio is the number obtained when we compare magnitudes."

Wallis, *Mathesis Universalis*:

> "When a comparison in terms of ratio is made, the resultant ratio often (namely with the exception of the 'numerical genus' itself) leaves the genus of quantities compared, and passes into the numerical genus, whatever the genus of quantities compared may have been."

Newton, *Of the Measures of Ratios*:

> "Now *a Ratio is a certain Habitude of two Magnitudes with Regard to Quantity*. And *because this Habitude regards the Relation of Quantity alone* exclusive of Circumstances of Forms and Species, it comes to pass that *it can be expressed by no Method, but by Numbers*; to wit, by the most general Ideas of the Magnitudes themselves."

Wallis elaborates:

> [The] whole definition of λόγος (*Ratio, Rate*, or *Proportion*) . . . [is] *that Relation of two Homogeneous Magnitudes* (or Magnitudes of the same kind,) *how the one stands related to the other, as to the* (Quotient, or) *Quantuplicity*: That is, *How many times*, (or How much of a time, or times,) *one of them contains the other*. The *English* word *How-many-fold*, doth in part answer it, . . . but because beside these which are properly called *Multiple* or *Many-fold*, (such as the *Double, Treble*, &c. which are denominated by whole Numbers,) there be many others to be denominated by Fractions, (proper or improper,) or Surds, or otherwise; . . . to which would answer (in *English*,) *How-much-fold*, (if we had such a word) . . .

Source: *Treatise of Algebra*. John Playford, London, 1685.

Hölder made this precise

Let K be a kind of magnitude. Suppose A and U are magnitudes of K. According to Euclid *Elements*, Book V, Definition 5, the ratio $A\!:\!U$ is completely characterized by the set:

$$\{\,(m,n) \in \mathbb{N} \times \mathbb{N} \mid mU < nA\,\},$$

which in turn is completely determined by the bounded lower set:

$$\{\,m/n \in \mathbb{Q} \mid mU < nA\,\}.$$

Accordingly, let us define:

$$[\,A\!:\!U\,] := \sup\{\,m/n \in \mathbb{Q} \mid mU < nA\,\},$$

and call this "the measure of A by U" or "the ratio of A to U."

“Axioms for Magnitude and the Theory of Measurement”

In his 1901 paper, Hölder stated the axioms for magnitudes very precisely, and showed that if K is any kind of magnitude and U is a magnitude of kind K, then the map

$$A \mapsto [\,A\!:\!U\,] : K \to \mathbb{R}$$

preserves order and addition, and its fibers are the equivalence classes in K.

It turns out that the assumptions about magnitudes described above and stated precisely by Hölder are equivalent to the assertion that $\overline{K}$ is a subsemigroup of the set of positive elements in a totally ordered archimedean group. (Demonstrating this does not involve any appeal to facts about the real numbers. The theorem below *does*.)

Hölder's Theorem and its Modern Influence

Hölder's Theorem. Every archimedean totally-ordered group is order-isomorphic to a sub-group of the additive real numbers.

Hölder's Theorem was a seminal result for much research on ordered algebraic structures in the 20^{th} century. Important generalizations include:

- Hahn Representation Theorem for totally-ordered abelian groups,
- Yosida's Representation Theorem for vector lattices,
- the Conrad-Harvery-Holland Representation Theorem for abelian lattice-ordered groups,
- the Localic Yosida Theorem (Madden, 1991).

Hölder's Theorem: Extensions relevant to measurement

Theorem. (Change of Unit). Suppose G is an archimedean totally-ordered group and $a, u, w \in G$ and u, w strictly positive. Then

$$[\,a\!:\!w] = [\,a\!:\!u\,][\,u\!:\!w\,].$$

Theorem. (Products). Suppose G, H and K are archimedean totally-ordered groups and

$$(g, h) \mapsto g \cdot h : G \times H \to K$$

is an order-homomorphism in each variable separately. Then for any $a, u \in G$ and $b, v \in H$, with u and v strictly positive:

$$[\,a \cdot b\!:\!u \cdot v\,] = [\,a\!:\!u\,][\,b\!:\!v\,].$$

For proofs, see: J. Madden. (2008.) *Ratio and Proportion in Euclid.* Available at: http://commoncoretools.me/wp-content/uploads/2015/07/Proportion-Euclid-Madden.pdf

Summary of main points: History of Ideas

- From the time of Euclid, through the Scientific Revolution, scholarly mathematicians viewed magnitudes as a separate ontological category, distinct from number, capable of being reasoned about without number, but measurable *by* number.

- Even today, physical scientists view the objects they study not as numbers but as non-numerical magnitudes. Physicists reason "proportionally" about magnitudes without mapping them to numbers.

- Mathematics in the Hindu-Arabic tradition is number-centered. Modern-practical mathematics follows this tradition, leaving non-numerical quantities in the shadows.

Summary of main points: Pedagogy

- In the modern K-12 mathematics curriculum, the connections of numbers to things are often implicit, incidental, unexamined and unexplained.
- Textbooks mention *quantities*, such as "26 miles," and call for competence in reasoning with them, e.g., by "dimensional analysis," but leaves them in ontological limbo. CCSSM calls quantities "numbers with units, which involves measurement." This is not a *category of being*.
- Measurement is naturally associated with ratio and proportion, but current curricular plans do not appear to build on this.
- Some mathematicians seem to suggest that the Euclidean/Newtonian paradigm is dated and irrelevant due to our modern understanding of number. Our concept of number is indeed more flexible, but the issue lies elsewhere. . . :

The Grand Challenge:

How do we make the connections of

numbers to things

explicit, systematic, rigorous and

more easily grasped by young learners?

Supplementary Slides (Common Core 1)

The word "quantity" is multivocal in CCSSM.

- "Students use numbers, including written numerals, to represent quantities [non-numbers?] and to solve quantitative problems, such as counting objects in a set" (CCSS, Kindergarten, page 9).
- "Understand that each successive number name refers to a quantity [number?] that is one larger" (CCSS, Kindergarten, page 11).
- "Understand a fraction $1/b$ as the quantity [number?] formed by 1 part when a whole is partitioned into b equal parts..." (CCSS, 3rd grade, page 24).
- "Make tables of equivalent ratios relating quantities [non-number?] with whole-number measurements..." (CCSS, 6th grade, page 42).

Supplementary Slides (Common Core 2)

CCSSM says the following concerning "quantity":

> "In real world problems, the answers are usually not numbers but quantities: numbers with units, which involves measurement. In their work in measurement up through Grade 8, students primarily measure commonly used attributes such as length, area, and volume. In high school, students encounter a wider variety of units in modeling, e.g., acceleration, currency conversions, derived quantities such as person-hours and heating degree days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages. ..." (CCSSM, High School, page 58).

Supplementary Slides (Common Core 3)

CCSSM: High School: Number and Quantity: Quantities*

Reason quantitatively and use units to solve problems.

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.
2. Define appropriate quantities for the purpose of descriptive modeling.
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

Supplementary Slides (Euclid on proportion)

A note on Euclid's definition of proportion

Although Euclidean ratios are relationships between magnitudes of the same kind, Euclid can compare a ratio between things of one kind to a ratio between things of another. The famous criterion for sameness of ratio is given in Definition 5 of Book V, and is recognized as a precursor of the modern definition of real number. The term that Euclid used to describe equal ratios is ἀνάλογον, which is translated into English as "in proportion." The term is introduced in Definition 6: "Magnitudes which have the same ratio are said to be in proportion." Having just referred in Definition 5 to four magnitudes forming two ratios and presented the criterion for these ratios to be the same, we may assume an identical context for Definition 6. *When the ratio of A to B is the same as the ratio of C to D, we say that the magnitudes form a* ***proportion****.*

Supplementary Slides (Mathematicians on ratio 1)

A modern mathematician defines "ratio":

> "Ratios are essentially just fractions, and understanding and working with ratios and proportions really just involves understanding and working with multiplication, division, and fractions. . . . To say that two quantities are in a ratio A to B means that for every A units of the first quantity there are B units of the second quantity."

Supplementary Slides (Mathematicians on ratio 2)

Another modern mathematician defines "ratio":

> "By definition, given two . . . [numbers] A and B, where $B \neq 0$ and both refer to the same unit (i.e., they are points on the same number line), the ratio of A to B, sometimes denoted by $A\colon B$, is the . . . [number] A/B."

Supplementary Slides (Mathematicians on ratio 3)

Yet another modern mathematician defines "ratio":

> "We say that the ratio between two quantities is $A\!:\!B$ if there is a unit so that the first quantity measures A units and the second measures B units. ... Two ratios are equivalent if one is obtained from the other by multiplying or dividing all the measurements by the same nonzero number. ... A proportion is a statement that two ratios are equal."

Supplementary Slides (Mathematicians on ratio 4)

Comments on how modern mathematicians define "ratio."

In one way or another, they all say that we form a ratio out of a pair of numbers, or that a ratio is nothing but a pair of numbers. The things themselves—what Euclid calls **magnitudes**—are not mentioned. If we take these statements seriously, the term "ratio" is not essential part of mathematical vocabulary, but rather it is a word used to signal that the numbers that are involved originate as the measures of two things whose relationship is of concern.

The passages betray the notion that the vocabulary of mathematics includes words for numbers, for sets of numbers, for indexed arrays of numbers, for relationships between numbers and for operations on numbers, but does not include words that refer to quantities that are *not* made out of numbers.

Supplementary Slides (Magnitude in physics 1)

Physicists reason about magnitudes.

The quantities of physics are not labeled numbers but magnitudes much as conceptualized by Euclid. The basic magnitudes are length, mass, and time, and other magnitudes are composites of the basic magnitudes, e.g., velocity is length/time, acceleration is velocity/time, force is mass $\cdot$ acceleration, energy is length $\cdot$ force, and power is energy/time.

If a unit is chosen for each basic magnitude, then each instance of each magnitude has an associated number. But in physics, it is more productive to reason with the magnitudes than with the numbers assigned to them through a choice of unit, or so says physicist Sanjoy Mahajan, author of [*Street-Fighting Mathematics: The Art of Educated Guessing and Opportunistic Problem Solving*. MIT Press, Cambridge, 2010].

Supplementary Slide (Magnitude in physics 2)

What Mahajan says about magnitude [ibid., p.4.]:

The inclusion of units, such as feet or feet per second in a problem about a falling body, he explains, "creates a significant problem. Because [if we are given that] the height is h feet, the variable h does not contain the units of height: h is therefore dimensionless." If the other variables in the given problem are also numerical, then they are also dimensionless, and likewise any combination of them is dimensionless. Consequently, no combination is favored. However, the kinds of the given quantities can guide us—and they will if we use variables to stand directly for magnitudes.

Supplementary Slide (Magnitude in physics 3)

What Mahajan says (cont.):

We should not pose the problem of a falling body by asking for "the number v of feet per second that the body is moving after falling h feet, given the acceleration a in feet per sec^2." Instead, we should understand each variable to stand for a quantity with a kind (or "dimension"), and we should recognize that we may only combine and compare magnitudes in a manner that is consistent with their kinds. We benefit thereby, because the physical meaning is built in to the terms with which we reason. If we ask, "What is the velocity v after falling a distance h, given the acceleration a," then evidently, the only magnitude we can compound from h and a that has the same genus as v is the square root of $h \cdot a$, and so we can expect v to be proportional to the square root of $h \cdot a$. We double the velocity by increasing the height by a factor of 4.

Supplementary Slide (A Challenge)

Of course, the ability to put "numbers-with-units" into formulae and "cancel units" at the appropriate times is important. But shouldn't we teach more than the mechanics? What is the *explanation* for a cancellation such as the following?

$$10000\,\cancel{\text{feet}} \times \frac{0.3048\,\text{meters}}{1\,\cancel{\text{foot}}} = 3048\,\text{meters}$$

Perhaps you think that the words are just decorations to remind us where the numbers came from. Or perhaps you prefer think of the words as symbols for magnitudes that are here being multiplied by numbers. *In either case, why is it that this cancellation procedure, which we have validated previously for numbers, can be used here, to operate on non-numerical symbols?*

Challenge: Provide a complete rigorous answer that could be grasped in a 7^{th}-grade classroom.